Mohyud-Din *et al. Asia Pac. J. Comput. Engin. (2015) 2:2*
DOI 10.1186/s40540-015-0014-y

Asia Pacific Journal on Computational Engineering
a SpringerOpen Journal

RESEARCH Open Access

Traveling wave solutions of Zakharov–Kuznetsov-modified equal-width and Burger's equations via $\exp(-\varphi(\eta))$-expansion method

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Abstract
In this article, a technique is proposed for obtaining better and accurate results for nonlinear PDEs. We constructed abundant exact solutions via $\exp(-\varphi(\eta))$-expansion method for the Zakharov–Kuznetsov-modified equal-width (ZK-MEW) equation and the $(2+1)$-dimensional Burgers equation. The traveling wave solutions are found through the hyperbolic functions, the trigonometric functions and the rational functions. The specified idea is very pragmatic for PDEs, and could be extended to engineering problems.

Keywords: Exp$(-\varphi(\eta))$-expansion method, Nonlinear evolution equation, (ZK-MEW) equation, Burger's equation, Solitary wave solutions

Background

Over the past few decades, researchers have shown keen interest in the solutions of nonlinear partial differential equations (PDEs).In the study of nonlinear physical phenomena, the investigation of solitary wave solutions [1–44] of nonlinear wave equations shows an important role. Scientific problems arise nonlinearly in numerous fields of mathematical physics, such as fluid mechanics, plasma physics, solid-state physics and geochemistry. Due to exact interpretation of nonlinear phenomena, these problems have gained much importance. However, in recent years, a variety of effective analytical methods has been developed to study soliton solutions of nonlinear equations, such as Backlund transformation method [1], tanh method [2–6], extended tanh method [7–12], pseudo-spectral method [13], trial function [14], sine–cosine method [15], Hirota method [16], exp function method [17–25], (G'/G)-expansion method [26–30], homogeneous balance method [31, 32], F-expansion method [33–35] and Jacobi elliptic function expansion method [36–38]. Ma et al. [39–44] established the complexiton solutions for Toda lattice equation. The theme of the method is that the exact solutions of nonlinear evolution equations can be articulated by $\exp(-\varphi(\eta))$, where $\varphi(\eta)$ gratifies the ordinary differential equation (ODE):

$$\left(\varphi'(\eta)\right) = \exp(-\varphi(\eta)) + \mu\ \exp(\varphi(\eta)) + \lambda \tag{1}$$

where $\eta = x - Vt$.

Explanation of $\exp(-\varphi(\eta))$-expansion method

Now, the $\exp(-\varphi(\eta))$-expansion method will be explained for constructing traveling wave solutions. Consider the general nonlinear partial differential equation for $u(x,t)$ is given by,

$$\phi(u, u_t, u_x, u_{tt}, u_{xx}, u_{xxx}, \ldots) = 0, \tag{2}$$

where $u(\eta) = u(x,t), \phi$ is a polynomial of u and its derivatives. Solving (2), the following steps are as.

Step 1 We Combine the variables by η,

$$u = u(\eta), \quad \eta = x - Vt, \tag{3}$$

where V is the speed of wave. Using Eqs. (3, 2) reduced to the following ODE for $u = u(\eta)$

$$G\left(u, u', u'', u''', u'''', \ldots\right) = 0, \tag{4}$$

Step 2 The solution of Eq. (4) can be articulated as

$$u(\eta) = \sum_{n=0}^{M} a_n \, (\exp(-\varphi(\eta)))^{\mathrm{n}}, \tag{5}$$

where $a_n 0 \leq n \leq M$ are constants such that $a_n \neq 0$ and $\varphi(\eta)$ satisfies Eq. (1). Our solutions now depend on the parameters involved in (1).

Family 1: When $\lambda^2 - 4\mu > 0$, we have

$$\varphi(\eta) = \ln\left\{\frac{1}{2\mu}\left(-\sqrt{(\lambda^2 - 4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\eta + c_1)\right) - \lambda\right)\right\}. \tag{6}$$

Family 2: When $\lambda^2 - 4\mu < 0$, we have

$$\varphi(\eta) = \ln\left\{\frac{1}{2\mu}\left(\sqrt{(\lambda^2 - 4\mu)}\tan\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\eta + c_1)\right) - \lambda\right)\right\}. \tag{7}$$

Family 3: When $\lambda^2 - 4\ \mu > 0\ \ \mu = 0$ and $\lambda \neq 0$,

$$\varphi(\eta) = -\ln\left\{\frac{\lambda}{\exp\left(\lambda(\eta + k)\right) - 1}\right\}. \tag{8}$$

Family 4: When $\lambda^2 - 4\ \mu = 0,\ \lambda \neq 0$, and $\mu \neq 0$,

$$\varphi(\eta) = \ln\left\{\frac{2(\lambda(\eta + k) + 2)}{\left(\lambda^2(\eta + k)\right)}\right\}. \tag{9}$$

Family 5: When $\lambda^2 - 4\ \mu = 0,\ \lambda = 0$, and $\mu = 0$,

$$\varphi(\eta) = \ln(\eta + k) \tag{10}$$

Step 3 By considering the homogenous principal, in Eq. (4). Considering Eqs. (1, 4, 5), we have $\mathrm{e}^{M\varphi(\eta)}$. We get algebraic equations with a_n, V, λ, μ, after comparing the same powers of $e^{\varphi(\eta)}$ to zero. We put the above values in Eq. (5) and with Eq. (1), we get some valuable traveling wave solutions of Eq. (2).

Solution procedure

Zakharov–Kuznetsov-modified equal-width equation

Consider the equation,

$$u_t + \alpha\left(u^n\right)_x + \left(\beta u_{xt} + \delta u_{yy}\right)_x = 0, \tag{11}$$

where α, β and δ are some nonzero parameters. We use $u = u(\eta), \eta = x + y - Vt$, we can convert Eq. (11) into an ODE.

$$-Vu' - \beta Vu''' + \delta u''' + 2\alpha uu' = 0, \tag{12}$$

where the dash denotes the derivative w. r. t. η. Now integrating Eq. (12), we have,

$$-Vu - \beta Vu'' + \delta u'' + \alpha u^2 + C = 0, \tag{13}$$

Using homogenous principle, balancing u'' and u^2, we have

$$2M = M + 2,$$
$$M = 2.$$

The trial solution of Eq. (12) can be stated as,

$$u(\eta) = a_2(\exp(-\varphi(\eta)))^2 + a_1(\exp(-\varphi(\eta))) + a_0, \tag{14}$$

where $a_2 \neq 0$, a_1 and a_0 are constants, while λ, μ are any constants.

Putting u, u', u'', u^2 in Eq. (13) and comparing, we get,

$$\begin{aligned}
&\alpha a_0^2 + \delta a_1\mu\lambda + C - 2\beta Va_2\mu^2 - \beta Va_1\mu\lambda + 2\delta a_2\mu^2 - Va_0 = 0,\\
&2\alpha a_0 a_1 + \delta a_1\lambda^2 + 2\delta a_1\mu + -2\beta Va_1\mu - 6\beta V\mu\lambda - \beta Va_1\lambda^2 + 6\delta a_2\mu\lambda - Va_1 = 0,\\
&2\alpha a_2 a_1 + 10\delta a_2\lambda + 2\delta a_1 + -2\beta Va_1 - 10\beta Va_2\lambda = 0,\\
&2\alpha a_2 a_1 + 10a_2\lambda + 2a_1 + -2\beta Va_1 - 10\beta Va_2\lambda = 0,\\
&\alpha a_2^2 + 6\delta a_2 - 6\beta Va_2 = 0,
\end{aligned} \tag{15}$$

By solving the algebraic equations, the required solution is given below.

$$\left\{V = \frac{1}{6}\frac{\alpha a_2 + 6\delta}{\beta}, \lambda = 0, a_0 = a_0, a_1 = 0, \mu = \frac{1}{2}\frac{1}{\beta\alpha a_2}\left(\sqrt{2}\sqrt{\beta\alpha\left(6C\beta + 6\alpha\beta a_0^2 - \alpha a_0 a_2 - 6a_0\delta\right)},\right)\right\}$$

where λ and μ are any constants.

Now putting the values in Eq. (14), we obtain

$$u = a_0 + a_2 e^{-2\varphi(\eta)}, \tag{16}$$

where $\eta = x - Vt$. By putting (6–10) in (16), we obtain the solutions which are given below.

Case 1 When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we have,

$$u_1(\eta) = a_0 + \frac{4a_2\mu^2}{\left(-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta + c_1)\right) - \lambda\right)^2},$$

where $\eta = x - \frac{1}{6}\frac{\alpha a_2 + 6\delta}{\beta}t$ and where c_1 is any constant.

Case 2 When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we have,

$$u_2(\eta) = a_0 + \frac{4a_2\mu^2}{\left(\sqrt{-\lambda^2 + 4\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\eta + c_1)\right) - \lambda\right)^2},$$

where $\eta = x - \frac{1}{6}\frac{\alpha a_2 + 6\delta}{\beta}t$ and where c_1 is any constant.

Case 3 When $\mu = 0$ and $\lambda \neq 0$, we have,

$$u_3(\eta) = a_0 + \frac{a_2\lambda^2}{\left(\exp(\eta + c_1)^\lambda - 1\right)^2},$$

where $\eta = x - \frac{1}{6}\frac{\alpha a_2 + 6\delta}{\beta}t$ and where c_1 is any constant.

Case 4 When $\lambda^2 - 4\mu = 0$, $\lambda \neq 0$, and $\mu \neq 0$, we obtain,

$$u_4(\eta) = a_0 + \frac{a_2(\eta + c_1)^2\lambda^4}{\left(2(\eta + c_1)^\lambda + 2\right)^2},$$

where $\eta = x - \frac{1}{6}\frac{\alpha a_2 + 6\delta}{\beta}t$ and where c_1 is any constant.

Case 5 When $\lambda = 0$, and $\mu = 0$, we have, $u_5(\eta) = a_0 + \frac{a_2}{(\eta + c_1)^2}$, where $\eta = x - \frac{1}{6}\frac{\alpha a_2 + 6\delta}{\beta}t$ and where c_1 is any constant.

Graphical demonstration

The graphs are given in Figs. 1, 2, 3, 4 and 5.

(2 + 1)-dimensional Burger's equation

Consider the equation,

$$u_t - uu_x - u_{xx} - u_{yy} = 0, \tag{17}$$

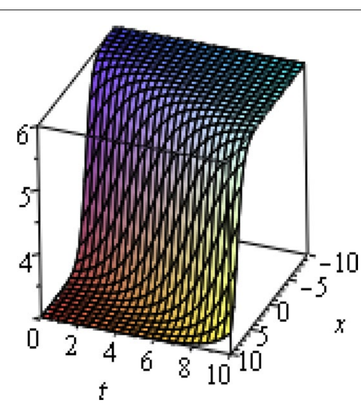

Fig. 1 Kink wave solution of u_1 when $a_2 = 1,\ a_0 = 2,\ y = 0,\ \lambda = 3,\ \mu = 2,\ c_1 = 1$

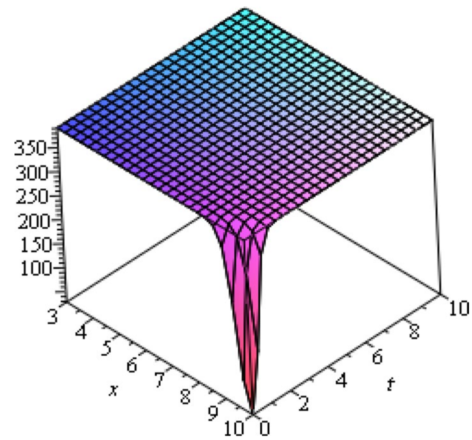

Fig. 2 Singular kink wave solution u_2 when $a_2 = 10,\ a_0 = 8,\ y = 0,\ \lambda = 7,\ \mu = 5,\ c_1 = -10$

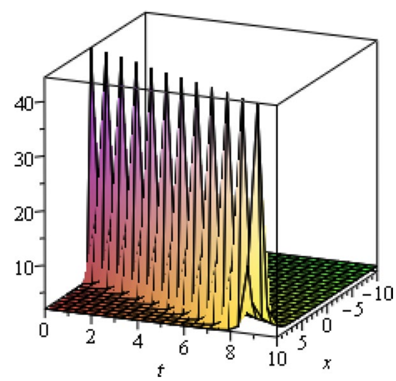

Fig. 3 Singular kink wave solution u_3 when $a_2 = 1, a_0 = 2, y = 0, \lambda = 1, c_1 = -1$

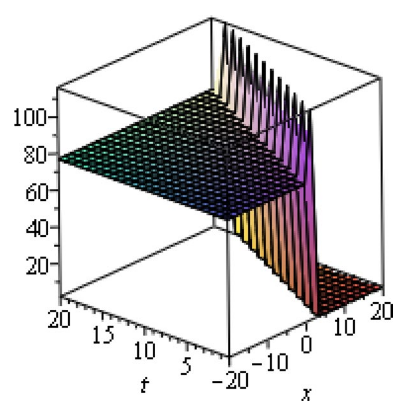

Fig. 4 Singular kink wave solution u_4 when $a_2 = 3, a_0 = 2, y = 0, \lambda = 5, \mu = 4, c_1 = -2$

where α, β and δ are some nonzero parameters. We have, $u = u(\eta)$, $\eta = x + y - Vt$, we can convert Eq. (17) into an ODE.

$$-Vu' - 2u'' - uu' = 0, \tag{18}$$

where dash denotes the derivative w. r. t.η.

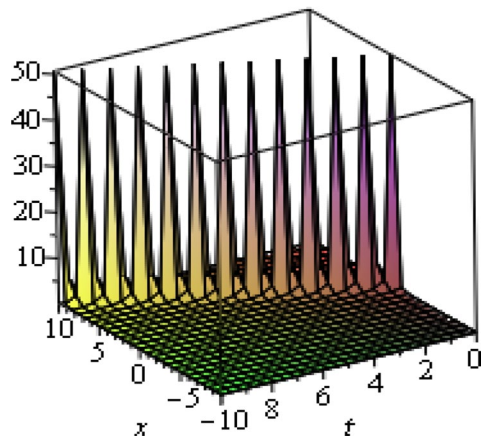

Fig. 5 Singular kink wave solution u_5 when $a_2 = 0.5, a_0 = 0.2, y = 0, \lambda = 0.1, c_1 = -0.1$

Integrating Eq. (18), we have,

$$-Vu - 2u' - \frac{1}{2}u^2 + C = 0, \tag{19}$$

Using homogenous principle, balancing the u' and u^2, we have, $M = 1$.

The trial solution of Eq. (18) can be stated as,

$$u(\eta) = a_1\ (\exp(-\varphi(\eta))) + a_0, \tag{20}$$

where $a_1 \neq 0$, a_0 is a constant, while λ, μ are any constants. By putting u, u', u'', u^2 in Eq. (19) and comparing, we get

$$\begin{aligned} &-\frac{1}{2}a_0^2 + 2a_1\mu + C - Va_0 = 0, \\ &-a_0a_1 + 2a_1\lambda - Va_1 = 0, \\ &-\frac{1}{2}a_1^2 + 2a_1 = 0, \end{aligned} \tag{21}$$

By solving the algebraic equations, the required solution is given below.

$$\left\{ \lambda = \tfrac{1}{2}\sqrt{V^2 + 2C + 16\mu}, a_0 = -V + \sqrt{V^2 + 2C + 16\mu}, \quad a_1 = 4, \right\}$$

where λ and μ are any constants. Now putting the values in Eq. (20), we obtain,

$$u = -\mathrm{V} + \sqrt{V^2 + 2C + 16\mu} + 4e^{-\varphi(\eta)}, \tag{22}$$

where $\eta = x - Vt$.

Now putting (6–10) in (22), we obtain the solutions as.

Case 1 When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we have,

$$u_6(\eta) = -1 + \sqrt{1 + 2C + 16\mu} + \frac{8\mu}{\left(-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\eta + c_1)\right) - \lambda\right)},$$

where $\eta = x - \mathrm{V}t$ and where c_1 is any constant.

Case 2 When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we obtain,

$$u_7(\eta) == -1 + \sqrt{1+2C+16\mu} + \frac{8\mu}{\left(+\sqrt{-\lambda^2+4\mu}\tanh\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\eta+c_1)\right) - \lambda\right)},$$

where $\eta = x - \mathrm{V}t$ and where c_1 is any constant.

Case 3 When $\mu = 0$ and $\lambda \neq 0$, we have,

$$u_8(\eta) === -1 + \sqrt{1+2C+16\mu} + \frac{4\lambda}{\left((\eta+c_1)^{\lambda} - 1\right)},$$

where $\eta = x - Vt$ and where c_1 is any constant.

Case 4 When $\lambda^2 - 4\mu = 0$, $\lambda \neq 0$, and $\mu \neq 0$, we obtain,

$$u_9(\eta) = -1 + \sqrt{1+2C+16\mu} + \frac{4(\eta+c_1)\lambda^2}{\left(2(\eta+c_1)^{\lambda} + 2\right)},$$

where $\eta = x - Vt$ and where c_1 is any constant.

Case 5 When $\lambda = 0$, and $\mu = 0$, we have,

$$u_{10}(\eta) = -1 + \sqrt{1+2C+16\mu} + \frac{4}{(\eta+c_1)},$$

where $\eta = x - Vt$ and where c_1 is any constant.

Graphical illustration

The graphs are given in Figs. 6, 7, 8, 9 and 10.

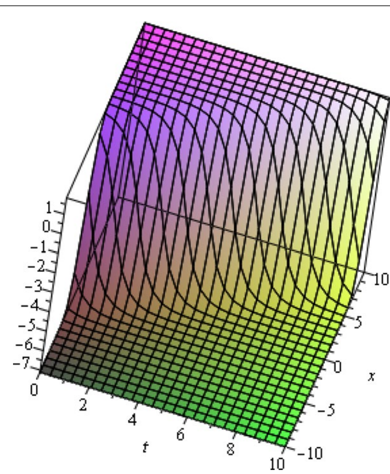

Fig. 6 Kink wave solution u_6 when $C = 1, a_0 = 1, y = 0, \lambda = 3, \mu = 1, c_1 = 1$

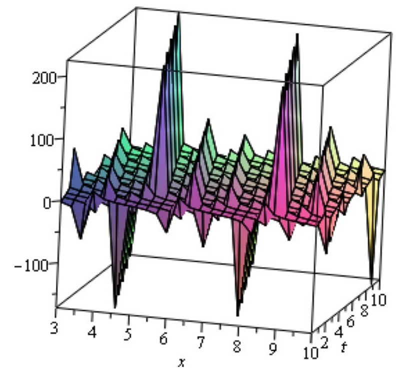

Fig. 7 Periodic solution $u_7(\eta)$ when $a_2 = 2, C = 1, y = 0, \lambda = 1, \mu = 2, c_1 = -1$

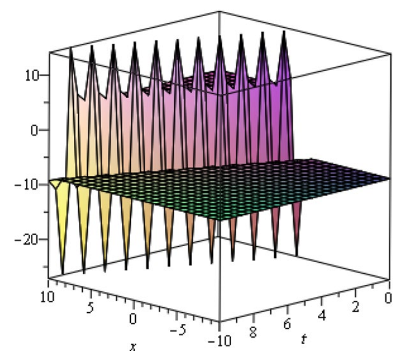

Fig. 8 Singular kink wave solution u_8 when $\mu = 1, C = 1, y = 0, \lambda = 3, c_1 = -1$

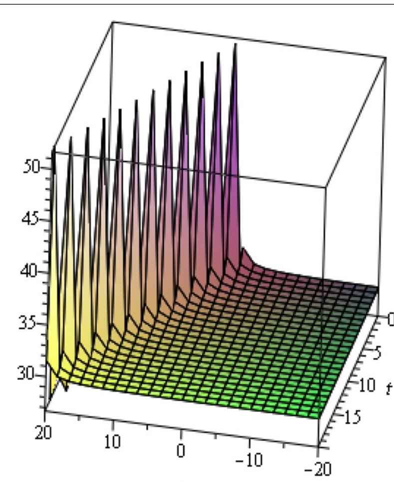

Fig. 9 Singular kink wave solution u_9 when $a_2 = 1, C = 1, y = 0, \lambda = 13, \mu = 1, c_1 = -1$

Conclusions

The $\exp(-\varphi(\eta))$-expansion method has been successfully applied to find the exact solutions of (ZK-MEW) equation and the Burger's equation. The attained results show that

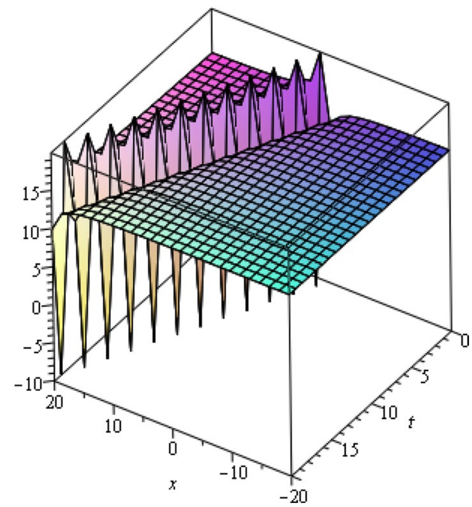

Fig. 10 Singular kink wave solution u_{10} when $C = 15, y = 0, \mu = 12, c_1 = -1$

the proposed technique is effective and capable for solving nonlinear partial differential equations. In this study, some exact solitary wave solutions, mostly solitons and kink solutions, are obtained through the hyperbolic and rational functions. This study shows that the proposed method is quite proficient and practically well organized in finding exact solutions of other physical problems.

Authors' contributions

The work was carried out in cooperation among all the authors (STM-D, AA and MAI). All authors have a good involvement to plan the paper, and to execute the analysis of this research work together. All authors read and approved the final manuscript.

Compliance with ethical guidelines

Competing interests

The authors declare that they have no competing interests.

Received: 1 June 2015 Accepted: 20 September 2015
Published online: 07 October 2015

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